

CTP TAMU-14/01 UPR-936-T MCTP-01-21

May 2001

hep-th/0105096

Massless 3-branes in M-theoryM. Cvetič^{†1}, H. Lü^{*2} and C.N. Pope^{‡3}[†]*Department of Physics and Astronomy**University of Pennsylvania, Philadelphia, PA 19104, USA*^{*}*Michigan Center for Theoretical Physics**University of Michigan, Ann Arbor, Michigan 48109, USA*[‡]*Center for Theoretical Physics**Texas A&M University, College Station, TX 77843, USA***ABSTRACT**

We construct supersymmetric M3-brane solutions in $D = 11$ supergravity. They can be viewed as deformations of backgrounds taking the form of a direct product of four-dimensional Minkowski spacetime and a non-compact Ricci-flat manifold of G_2 holonomy. Although the 4-form field strength is turned on it carries no charge, and the 3-branes are correspondingly massless. We also obtain 3-branes of a different type, arising as M5-branes wrapped over S^2 .

¹ Research supported in part by DOE grant DE-FG02-95ER40893 and NATO grant 976951.

² Research supported in full by DOE grant DE-FG02-95ER40899

³ Research supported in part by DOE grant DE-FG03-95ER40917.

1 Introduction

The standard D3-brane provides a natural supergravity dual of four dimensional $\mathcal{N} = 4$ superconformal Yang-Mills theory, *via* the AdS/CFT correspondence [1, 2, 3]. Branes with less supersymmetry can in general be constructed by replacing the spheres that form the level surfaces in the flat transverse space by some other Einstein space that admits a lesser number of Killing spinors [4]. It was proposed that D3-branes on the six-dimensional conifold, in which the level surfaces are the $T^{1,1}$ space, is dual to an $\mathcal{N} = 1$ superconformal theory in $D = 4$ with gauge group $SU(N) \times SU(N)$ [5]. The conformal symmetry can then itself be broken, by introducing fractional branes corresponding to the wrapping of D5-branes on 2-cycles. The corresponding supergravity solutions were obtained in [6, 7].

It has been proposed that M-theory compactified on a certain singular seven-dimensional space with G_2 holonomy might be related to a $\mathcal{N} = 1$, $D = 4$ gauge theory [8, 9, 10], which has no conformal symmetry to begin with. (See also the recent papers [11, 12, 13, 14, 15].) This leads to the question of whether there might exist a 3-brane configuration in M-theory, whose transverse space is a deformation of a Ricci-flat space of G_2 holonomy, in which the 4-form field is turned on. In this paper we shall indeed obtain 3-brane solutions of this deformed type.

So far, three explicit metrics for seven-dimensional manifolds of G_2 holonomy are known [16, 17]. They all have cohomogeneity one. The first two have principal orbits that are \mathbb{CP}^3 or $SU(3)/(U(1) \times U(1))$, written as an S^2 bundle over S^4 or \mathbb{CP}^2 respectively. The associated 7-manifolds have the topology of \mathbb{R}^3 bundles over S^4 or \mathbb{CP}^2 . The third manifold has principal orbits that are topologically $S^3 \times S^3$, written as an S^3 bundle over S^3 , and the 7-manifold is topologically $\mathbb{R}^4 \times S^3$. In order to construct a non-trivial 3-brane configuration on such a background in eleven-dimensional supergravity, it is necessary that the background G_2 manifold itself should admit a well-behaved harmonic 4-form (or dual 3-form). It was shown in [18, 19] that such harmonic forms exist in all three of these explicit examples. In this paper, we construct M3-brane configurations describing deformations away from backgrounds having a G_2 manifold as the transverse space, taking the 4-form field strength of M-theory to be proportional to the appropriately deformed harmonic 4-form. We first obtain the second-order equations for the fields in our ansatz, which follow from those of eleven-dimensional supergravity, and then we show that in a Lagrangian formulation of these equations the potential can be derived from a superpotential. This leads to first-order equations which we are able to solve explicitly.

The exact solutions that we obtain by this method describe configurations with a four-

dimensional Poincaré invariance in the world-volume, and a seven-dimensional transverse space that is a deformation of the original Ricci-flat metric of G_2 holonomy. We may thus view them as being 3-brane solutions of M-theory. At large distance they approach the product of 4-dimensional Minkowski spacetime and the Ricci-flat metric of G_2 holonomy. The rate at which the metrics approach this asymptotic form is rapid enough that the ADM mass vanishes, and so they may be thought of as massless M3-branes. In common with other examples of massless branes, they have naked singularities at short distance. We show that the M3-brane solutions are supersymmetric.

It is of interest also to look for 3-brane configurations in M-theory within a more general framework. Another natural candidate for a 3-brane is to look for an M5-brane wrapped on a supersymmetric 2-cycle. Wrapped supersymmetric M5-branes have been discussed in previous papers [30, 31], and typically these have been of the form $\text{AdS}_d \times H_{7-d}$, where H_n denotes the n -dimensional hyperbolic space. In section 5, we shall consider M5-branes wrapping around a 2-sphere. The solutions can be obtained by starting with $SU(2)$ -gauged AdS supergravity in $D = 7$, and looking for 3-branes supported by the Yang-Mills fields. We obtain the equations of motion for the general non-abelian case, and show that when only a $U(1)$ subgroup is turned on, we can construct first-order equations derivable from a superpotential. The general solution can be reduced to Abel's equation, and the structure of the resulting configurations can be analysed. The solutions can be lifted to $D = 11$, where they describe 3-branes as M5-branes wrapped on S^2 .

2 M3-branes in backgrounds of \mathbb{R}^3 bundles over S^4 or \mathbb{CP}^2

In this section we shall construct M3-brane solutions that can be viewed as living in backgrounds where the 7-dimensional transverse space is a manifold of G_2 holonomy with the topology of the \mathbb{R}^3 bundle over S^4 or \mathbb{CP}^2 .

2.1 The ansatz

Let us consider the $D = 11$ ansatz

$$ds_{11}^2 = H^2 dx^\mu dx_\mu + d\rho^2 + a^2 D\mu^i D\mu^i + b^2 d\Omega_4^2, \quad (1)$$

where $\mu^i \mu^i = 1$ and $D\mu^i = d\mu^i + \epsilon_{ijk} A_{(1)}^j \mu^k$, and $A_{(1)}^i$ is the $SU(2)$ Yang-Mills instanton on S^4 , whose unit metric is $d\Omega_4^2$. The functions H , a and b will be taken to depend only on the radial coordinate ρ in the transverse space. This describes the case of the \mathbb{R}^3 bundle over

S^4 . The second possibility is obtained by replacing the S^4 by \mathbb{CP}^2 . This does not affect the form of the equations for H , a and b .¹ The constrained μ^i coordinates can be expressed in terms of two angular coordinates on S^2 in a standard way,

$$\mu_1 = \sin \theta \sin \phi, \quad \mu_2 = \sin \theta \cos \phi, \quad \mu_3 = \cos \theta. \quad (2)$$

The vielbein components in the S^2 fibre directions are then given by

$$\begin{aligned} e^1 &= a(d\theta - A_{(1)}^1 \cos \phi + A_{(1)}^2 \sin \phi), \\ e^2 &= a \sin \theta (d\phi + A_{(1)}^1 \cot \theta \sin \phi + A_{(1)}^2 \cot \theta \cos \phi - A_{(1)}^3). \end{aligned} \quad (3)$$

There is clearly a vacuum solution of the form (1) that is simply the direct product of four-dimensional Minkowski spacetime and the associated Ricci-flat seven-dimensional manifold with G_2 holonomy. In the vacuum we shall have $H = 1$, with a and b being given by [16, 17]. We should now like to turn on the 4-form field strength of eleven-dimensional supergravity. The 4-form ansatz that respects the symmetry of the metric is given by [17, 19]

$$\begin{aligned} F_{(4)} &= f_1 \Omega_{(4)} + f_2 X_{(2)} \wedge Y_{(2)} + f_3 d\rho \wedge Y_{(3)}, \\ *F_{(4)} &= H^4 a^2 b^{-4} f_1 \epsilon_{(4)} \wedge d\rho \wedge X_{(2)} + H^4 a^{-2} f_2 \epsilon_{(4)} \wedge d\rho \wedge Y_{(2)} + H^4 f_3 \epsilon_{(4)} \wedge X_{(3)}, \end{aligned} \quad (4)$$

where the f_i are functions depending only on ρ , and

$$\begin{aligned} X_{(2)} &\equiv \frac{1}{2} \epsilon_{ijk} \mu^i D\mu^j \wedge D\mu^k, & Y_{(2)} &\equiv \mu^i F_{(2)}^i, & X_{(3)} &\equiv D\mu^i \wedge F_{(2)}^i, \\ Y_{(3)} &\equiv \epsilon_{ijk} \mu^i D\mu^j \wedge F_{(2)}^k, & \epsilon_{(4)} &\equiv dt \wedge dx_1 \wedge dx_2 \wedge dx_3. \end{aligned} \quad (5)$$

(Note that $F_{(4)}$ could in principle have had a term of the form $d\rho \wedge X_{(3)}$ as well, but this is ruled out by the field equation $d*F_{(4)} = 0$.) The Bianchi identity $dF_{(4)} = 0$ implies that $f'_1 = 4f_3$ and $f'_2 = 2f_3$, so we can take $f_1 = 2f$, $f_2 = f$ and $f_3 = \frac{1}{2}f'$, giving

$$F_{(4)} = f(2\Omega_{(4)} + X_{(2)} \wedge Y_{(2)}) + \frac{1}{2}f' d\rho \wedge Y_{(3)}. \quad (6)$$

In fact $F_{(4)} = dA_{(3)}$ with $A_{(3)} = \frac{1}{2}f Y_{(3)}$. The field equation $d*F_{(4)} = 0$ implies

$$(2a^4 + b^4) H^4 f - \frac{1}{2}a^2 b^4 (H^4 f')' = 0. \quad (7)$$

(The $F_{(4)} \wedge F_{(4)}$ term vanishes here.)

¹In everything that follows, results obtained for the case of the S^2 bundle over S^4 apply equally, *mutatis mutandis*, to the case of the S^2 bundle over \mathbb{CP}^2 .

In order to impose the $D = 11$ Einstein equation it is convenient to perform a Kaluza-Klein reduction on the 4-dimensional world volume of the 3-brane, so that the problem can be reformulated from a seven-dimensional point of view. This allows us to make use of curvature calculations for 7-metrics of this type that were performed in [17]. The relevant seven-dimensional Lagrangian is given by

$$e^{-1} \mathcal{L}_7 = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{48} e^{\sqrt{\frac{8}{5}}\phi} F_{(4)}^2, \quad (8)$$

with the ansatz (1) now taking the form

$$\begin{aligned} ds_7^2 &= dt^2 + \tilde{a}^2 D\mu^i D\mu^i + \tilde{b}^2 d\Omega_4^2, \\ F_{(4)} &= f(2\Omega_{(4)} + X_{(2)} \wedge Y_{(2)}) + \frac{1}{2} f' d\rho \wedge Y_{(3)}. \end{aligned} \quad (9)$$

The metric in $D = 7$ is related to the one in $D = 11$ by

$$d\hat{s}_{11}^2 = e^{-\frac{2}{3}\sqrt{\frac{2}{5}}\phi} ds_7^2 + e^{\frac{1}{3}\sqrt{\frac{5}{2}}\phi} dx^\mu dx_\mu. \quad (10)$$

Thus we have

$$dt = H^{4/5} d\rho, \quad \tilde{a} = H^{4/5} a, \quad \tilde{b} = H^{4/5} b. \quad (11)$$

The original eleven-dimensional Einstein equation is now recast as the seven-dimensional dilaton equation and Einstein equation. The dilaton equation gives

$$(a^2 b^4 (H^4)')' = \frac{2}{3} H^4 \left(f'^2 + \frac{2f^2}{a^2} + \frac{4f^2 a^2}{b^4} \right), \quad (12)$$

which can be rewritten as

$$\frac{6H''}{H} + \frac{18H'^2}{H^2} + \frac{12a' H'}{a H} + \frac{24b' H'}{b H} = \frac{f'^2}{a^2 b^4} + \frac{2f^2}{a^4 b^4} + \frac{4f^2}{b^8}. \quad (13)$$

From the Einstein equation we get three separate equations, namely

$$\begin{aligned} \frac{5a''}{a} + \frac{28a' H'}{a H} + \frac{5a'^2}{a^2} + \frac{20a' b'}{a b} + \frac{16b' H'}{b H} + \frac{4H''}{H} + \frac{12H'^2}{H^2} - \frac{5}{a^2} - \frac{5a^2}{b^4} &= \frac{f'^2}{4a^2 b^4} - \frac{2f^2}{a^4 b^4} + \frac{6f^2}{b^8}, \\ \frac{5b''}{b} + \frac{36b' H'}{b H} + \frac{15b'^2}{b^2} + \frac{10a' b'}{a b} + \frac{8a' H'}{a H} + \frac{4H''}{H} + \frac{12H'^2}{H^2} - \frac{15}{b^2} + \frac{5a^2}{b^4} &= \frac{f'^2}{4a^2 b^4} + \frac{f^2}{2a^4 b^4} - \frac{4f^2}{b^8}, \\ \frac{10a''}{a} + \frac{8a' H'}{a H} + \frac{20b''}{b} + \frac{16b' H'}{b H} + \frac{24H''}{H} + \frac{12H'^2}{H^2} &= -\frac{f'^2}{a^2 b^4} + \frac{3f^2}{a^4 b^4} + \frac{6f^2}{b^8}. \end{aligned} \quad (14)$$

This set of equations can be derived from the Lagrangian $L = T - V$, where

$$\begin{aligned} T &= -\frac{5\dot{a}^2}{2a^2} - \frac{20\dot{a}\dot{b}}{a b} - \frac{15\dot{b}^2}{b^2} - \frac{20\dot{a}\dot{H}}{a H} - \frac{40\dot{b}\dot{H}}{b H} - \frac{15\dot{H}^2}{H^2} + \frac{5\dot{f}^2}{8a^2 b^4}, \\ V &= \frac{5}{4} H^8 \left(2a^2 b^4 (-a^4 + b^4 - 6a^2 b^2) - (2a^4 + b^4) f^2 \right), \end{aligned} \quad (15)$$

together with the constraint $T + V = 0$. Note that here a dot is a derivative with respect to the coordinate η , defined by $d\rho = a^2 (bH)^4 d\eta$. The kinetic term T can be expressed as $T = \frac{1}{2} g_{ij} \dot{\alpha}_i \dot{\alpha}_j$, where $a = e^{\alpha_1}$, $b = e^{\alpha_2}$, $H = e^{\alpha_3}$ and $f = \alpha_4$, with

$$g_{ij} = \begin{pmatrix} -5 & -20 & -20 & 0 \\ -20 & -30 & -40 & 0 \\ -20 & -40 & -30 & 0 \\ 0 & 0 & 0 & \frac{5}{4a^2 b^4} \end{pmatrix}. \quad (16)$$

Since g_{ij} is field dependent, the system is a non-linear sigma-model. Nevertheless, we find that the potential V can be expressed in terms of a superpotential W , so that $V = -\frac{1}{2} g^{ij} \partial_i W \partial_j W$, where

$$W = \frac{5}{2} H^4 (a^2 + b^2) \sqrt{4a^2 b^4 + f^2}. \quad (17)$$

2.2 First-order equations and general solution

From the superpotential (17), we can derive the first-order equations $\dot{\alpha}^i = g^{ij} \partial_j W$, which, in terms of the original radial coordinate ρ of equation (1) become

$$\begin{aligned} a' &= \frac{6a^4 b^4 - 6a^2 b^6 + a^2 f^2 - 2b^2 f^2}{3a b^4 K}, & f' &= \frac{2(a^2 + b^2) f}{K}, \\ b' &= -\frac{12a^4 b^4 + 4a^2 f^2 + b^2 f^2}{6a^2 b^3 K}, & H' &= \frac{(a^2 + b^2) f^2 H}{3a^2 b^4 K}, \end{aligned} \quad (18)$$

where a prime denotes $d/d\rho$, and we have defined

$$K \equiv \sqrt{4a^2 b^4 + f^2}. \quad (19)$$

Solutions of these equations will necessarily satisfy the original second-order equations, implying that we shall have solutions of the $D = 11$ supergravity equations. One may note from (18) that

$$a b^2 H^3 f = \kappa, \quad (20)$$

where κ is a constant of integration.

In order to solve the first-order equations (18) it is helpful to define new hatted variables as follows:

$$a = H^{-1/2} \hat{a}, \quad b = H^{-1/2} \hat{b}, \quad f = H^{-3/2} \hat{f}. \quad (21)$$

At the same time, we introduce a new radial variable τ , defined by $d\tau = H^{1/2} d\rho$. The metric ansatz (1) now assumes the form

$$ds_{11}^2 = H^2 dx^\mu dx_\mu + H^{-1} (d\tau^2 + \hat{a}^2 D\mu^2 + \hat{b}^2 d\Omega_4^2). \quad (22)$$

The first-order equations (18) are considerably simplified, becoming

$$\begin{aligned}\frac{d\hat{a}}{d\tau} &= \frac{(\hat{a}^2 - \hat{b}^2) \hat{K}}{2\hat{a} \hat{b}^4}, & \frac{d\hat{b}}{d\tau} &= -\frac{\hat{K}}{2\hat{b}^3}, \\ \frac{1}{\hat{f}} \frac{d\hat{f}}{d\tau} &= \frac{(\hat{a}^2 + \hat{b}^2) \hat{K}}{2\hat{a} \hat{b}^4}, & \frac{1}{H} \frac{dH}{d\tau} &= \frac{(\hat{a}^2 + \hat{b}^2) \hat{f}^2}{3\hat{a}^2 \hat{b}^4 \hat{K}},\end{aligned}\quad (23)$$

where $\hat{K} \equiv \sqrt{4\hat{a}^2 \hat{b}^4 + \hat{f}^2}$.

A further change of radial coordinate to r , defined by $dr = -\hat{K} d\tau$, puts these equations in the form

$$\begin{aligned}\frac{d\hat{a}}{dr} &= -\frac{(\hat{a}^2 - \hat{b}^2)}{2\hat{a} \hat{b}^4}, & \frac{d\hat{b}}{dr} &= \frac{1}{2\hat{b}^3}, \\ \frac{1}{\hat{f}} \frac{d\hat{f}}{dr} &= -\frac{(\hat{a}^2 + \hat{b}^2)}{2\hat{a} \hat{b}^4}, & \frac{1}{H} \frac{dH}{dr} &= -\frac{(\hat{a}^2 + \hat{b}^2) \hat{f}^2}{3\hat{a}^2 \hat{b}^4 (4\hat{a}^2 \hat{b}^4 + \hat{f}^2)}.\end{aligned}\quad (24)$$

In particular, the equations for \hat{a} and \hat{b} have now decoupled from the rest, and they are in fact nothing but the first-order equations for the G_2 metrics on the \mathbb{R}^3 bundle over S^4 (see, for example, [20]).

The system of first-order equations is now completely solvable. After a final change of radial variable $r \rightarrow \frac{1}{2}r^4$, we find that the general solution for the metric in the ansatz (1) is

$$ds_{11}^2 = H^2 dx^\mu dx_\mu + 2H^{-7} U^{-1} dr^2 + \frac{1}{2}r^2 H^{-1} U D\mu^i D\mu^i + r^2 H^{-1} d\Omega_4^2, \quad (25)$$

where

$$U = 1 - \frac{\ell^4}{r^4}, \quad H = \left(1 + \frac{c^2}{2r^{12} U^2}\right)^{1/6}, \quad (26)$$

and c is a constant. The function f appearing in the 4-form ansatz (6) is given by

$$f = \frac{c}{r^3 H^{3/2} U^{1/2}}. \quad (27)$$

Note that the constant κ appearing in (20) is precisely the constant c .

2.3 Properties of the 3-brane solution

The general solution contains two non-trivial integration constants, ℓ and c . The constant ℓ measures the scale size of the “gravitational instanton” of the background G_2 manifold. The constant c , on the other hand, measures the strength of the the 4-form field $F_{(4)}$. Asymptotically at large distance, the gravitational instanton contributions, going as ℓ^4/r^4 , dominate in comparison to the $F_{(4)}$ contribution, going as c^2/r^{12} .

When the constant c is set to zero, the 4-form field is turned off, and consequently the configuration reduces to the vacuum solution of a direct product $M_4 \times \mathcal{M}_7$ of Minkowski

4-spacetime and the seven-dimensional smooth manifold with G_2 holonomy. When c is non-vanishing, the solution describes a 3-brane in $D = 11$, with a four-dimensional Poincaré symmetry in its world volume. Asymptotically, the solution approaches $M_4 \times \mathcal{M}_7$. If the parameter ℓ is taken to be zero, then the smooth 7-manifold \mathcal{M}_7 has a singular limit to the cone over the S^2 bundle over S^4 (or \mathbb{CP}^2). At small distance, near $r = \ell$, the metric becomes singular, with the limiting form

$$\begin{aligned} \ell = 0 : \quad ds_{11}^2 &= \rho^{-\frac{1}{2}} dx^\mu \wedge dx_\mu + \rho^{\frac{1}{2}} \left(\frac{1}{2} D\mu^i D\mu^i + d\Omega_4^2 \right) + d\rho^2, \\ \ell \neq 0 : \quad ds_{11}^2 &= \rho^{-\frac{2}{5}} dx^\mu \wedge dx_\mu + \frac{1}{2} \rho^{\frac{1}{5}} D\mu^i D\mu^i + \rho^{\frac{4}{5}} d\Omega_4^2 + d\rho^2 \end{aligned} \quad (28)$$

as the proper distance ρ tends to zero. We can also consider the possibility of sending $\ell^4 \rightarrow -\ell^4$. In this case, the metric behaviour at small proper distance (i.e. near $r = 0$) becomes

$$ds_{11}^2 = \rho^{-1/4} (dx^\mu dx_\mu + \frac{1}{2} D\mu^i D\mu^i) + \rho^{1/2} d\Omega_4 + d\rho^2. \quad (29)$$

The 4-form flux is given by

$$Q = 2f \int_{S^4} \Omega_{(4)}, \quad (30)$$

and from (27) this can be seen to vanish when r is sent to infinity. Thus our M3-brane configurations do not carry any conserved charge, and might be described as “3-branes without 3-branes.” The solution should really be thought of as a gravitational monopole involving the supergravity multiplet. The two constants ℓ and c are both continuous parameters.

What is perhaps the most interesting feature of the solution is that it is massless. The leading-order r dependence in the function H at large r is c^2/r^{12} , whilst a mass term for a 7-dimensional transverse space would have a leading-order r -dependence of the form m/r^5 . The existence of the naked singularity may be related to the fact that the solution is massless. In fact all the previously known massless p -brane solutions in supergravity contain naked singularities [21, 22, 23, 24, 25]. A repulson mechanism was proposed in string theory [26] to resolve such a naked singularity in the massless dyonic string [25]. A further observation is there appears not to exist a natural and non-trivial decoupling limit. This may also be a consequence of the masslessness. For example, a massive dyonic string has a decoupling limit, but this ceases to exist when the charges are tuned for masslessness.

Unlike the dyonic string, where the masslessness is achieved by making an adjustment of integration constants, in our new M3-brane solution there *is* no mass integration constant. From the point of view of supersymmetry, the masslessness is consistent with the absence of a non-vanishing conserved 4-form charge. However, we shall defer a more detailed investigation of the supersymmetry of this solution until section 4.

3 M3-brane in background of \mathbb{R}^4 bundle over S^3

In this section we construct an analogous 3-brane solution in the background of the third manifold of G_2 holonomy, whose topology is $\mathbb{R}^4 \times S^3$. As we shall see, the configuration is again a “no-braner,” which carries no conserved brane charge.

3.1 The ansatz

In this case, we consider the eleven-dimensional metric ansatz

$$ds_{11}^2 = H^2 dx^\mu dx_\mu + d\rho^2 + a^2 \nu_i^2 + b^2 \Sigma_i^2 \quad (31)$$

where $\nu_i \equiv \sigma_i - \frac{1}{2}\Sigma_i$, and σ_i and Σ_i are two sets of left-invariant on two independent $SU(2)$ group manifolds. The level surfaces $r = \text{constant}$ are therefore an S^3 bundle over S^3 . Since the bundle is a trivial one, the level surfaces are topologically $S^3 \times S^3$. There is a vacuum solution which is a complete Ricci-flat manifold, namely the direct product of four-dimensional Minkowski spacetime and the known seven-dimensional manifold $\mathbb{R}^4 \times S^3$ with G_2 holonomy [16, 17].

Again we should now like to turn on the 4-form field strength, in order to introduce a 3-brane configuration in this background. The ansatz for the 4-form, respecting the symmetries of the vacuum, can be written as [18]

$$F_{(4)} = f_1 \nu_i \wedge \nu_j \wedge \Sigma_i \wedge \Sigma_j + f_2 d\rho \wedge \nu_1 \wedge \nu_2 \wedge \nu_3 + \frac{1}{2} f_3 \epsilon_{ijk} d\rho \wedge \nu_i \wedge \Sigma_j \wedge \Sigma_k. \quad (32)$$

The Bianchi identity $dF_{(4)} = 0$ gives

$$f_1' - \frac{1}{8}f_2 + \frac{1}{2}f_3 = 0, \quad (33)$$

and the field equation $d\hat{*}F_{(4)} = 0$ gives

$$\frac{2f_1 H^4}{a b} + (f_3 a b^{-1} H^4)' = 0, \quad -\frac{3f_1 H^4}{a b} + (2f_2 b^3 a^{-3} H^4)' = 0. \quad (34)$$

It is again convenient to derive the conditions implied by the eleven-dimensional Einstein equation in terms of a dimensional reduction to $D = 7$. The dilaton equation gives

$$\frac{H''}{H} + \frac{3H'^2}{H^2} + \frac{3a' H'}{a H} + \frac{3b' H'}{b H} - \frac{2f_1^2}{a^4 b^4} - \frac{f_2^2}{6a^6} - \frac{f_3^2}{2a^2 b^4} = 0, \quad (35)$$

and finally, the seven-dimensional Einstein equation gives

$$\begin{aligned} \frac{5b''}{b} + \frac{4H''}{H} + \frac{15a' b'}{a b} + \frac{10b'^2}{b^2} + \frac{12a' H'}{a H} + \frac{32b' H'}{b H} + \frac{12H'^2}{H^2} - \frac{5}{2b^2} + \frac{5a^2}{16b^4} + \frac{2f_1^2}{a^4 b^4} - \frac{3f_2^2}{2a^6} + \frac{f_3^2}{2a^2 b^4} &= 0, \\ \frac{5a''}{a} + \frac{4H''}{H} + \frac{15a' b'}{a b} + \frac{10a'^2}{a^2} + \frac{32a' H'}{a H} + \frac{12b' H'}{b H} + \frac{12H'^2}{H^2} - \frac{5}{2a^2} - \frac{5a^2}{32b^4} + \frac{2f_1^2}{a^4 b^4} + \frac{f_2^2}{a^6} - \frac{2f_3^2}{a^2 b^4} &= 0, \\ \frac{15a''}{a} + \frac{15b''}{b} + \frac{24H''}{H} + \frac{12a' H'}{a H} + \frac{12b' H'}{b H} + \frac{12H'^2}{H^2} - \frac{18f_1^2}{a^4 b^4} + \frac{f_2^2}{a^6} + \frac{3f_3^2}{a^2 b^4} &= 0. \end{aligned} \quad (36)$$

3.2 First order equations and general solution

From (34), we can solve for f_1 and f_2 ,

$$f_1 = -\frac{1}{2}a b H^{-4}(f_3 a b^{-1} H^4)', \quad f_2 = -\frac{3}{4}a^3 b^{-3} H^{-4}(\lambda + f_3 a b^{-1} H^4), \quad (37)$$

where λ is a constant of integration. The remaining equations for a , b , H and f_3 can then be obtained from the Lagrangian $L = T - V$, together with the constraint $T + V = 0$, where $T = \frac{1}{2}g_{ij} \dot{\alpha}^i \dot{\alpha}^j$ with $\alpha^i = (\log a, \log b, \log H, f_3)$, and a dot denotes a derivative with respect to η defined by $dt = a^3 b^3 H^4 d\eta$. We have

$$g_{ij} = \begin{pmatrix} -60 - \frac{15f_3^2}{b^4} & -90 + \frac{15f_3^2}{b^4} & -120 - \frac{60f_3^2}{b^4} & -\frac{15f_3}{b^4} \\ -90 + \frac{15f_3^2}{b^4} & -60 - \frac{15f_3^2}{b^4} & -120 + \frac{60f_3^2}{b^4} & \frac{15f_3}{b^4} \\ -120 - \frac{60f_3^2}{b^4} & -120 + \frac{60f_3^2}{b^4} & -120 - \frac{240f_3^2}{b^4} & -\frac{60f_3}{b^4} \\ -\frac{15f_3}{b^4} & \frac{15f_3}{b^4} & -\frac{60f_3}{b^4} & -\frac{15}{b^4} \end{pmatrix}. \quad (38)$$

The potential V is given by

$$V = \frac{15}{32}H^8 a^4 (-a^4 b^2 + 16a^2 b^4 + 16b^6 + 3a^4 b^{-2} f_3^2 + 16b^2 f_3^2) + \frac{45}{32}\lambda a^6 (\lambda - 2a b^{-1} H^4 f_3). \quad (39)$$

As in the previous case, the kinetic term T is of the form of a non-linear sigma model, with a fairly complicated field-dependent g_{ij} . The inverse is relatively simpler, given by

$$g^{ij} = \begin{pmatrix} \frac{1}{45} & -\frac{1}{90} & -\frac{1}{90} & \frac{1}{90}f_3 \\ -\frac{1}{90} & \frac{1}{45} & -\frac{1}{90} & \frac{7}{90}f_3 \\ -\frac{1}{90} & -\frac{1}{90} & \frac{1}{72} & -\frac{1}{18}f_3 \\ -\frac{1}{90}f_3 & \frac{7}{90}f_3 & -\frac{1}{18}f_3 & -\frac{3}{45}b^4 + \frac{13}{45}f_3^2 \end{pmatrix} \quad (40)$$

If the integration constant λ is taken to be zero,² we find that the potential can be expressed in terms of a superpotential W , *i.e.* $V = -\frac{1}{2}g^{ij} \partial_i W \partial_j W$, with

$$W = \frac{15}{4}H^4 a^2 b^{-1} (a^2 + 4b^2) \sqrt{b^4 - f_3^2}. \quad (41)$$

²If the integration constant λ were non-vanishing, which would correspond to a configuration including M5-branes wrapped on 3-cycles in $S^3 \times S^3$, it is not clear how one would solve the second-order equations. Similar remarks apply to the previous case in section 2 also. We chose to omit an analogous constant of integration, in the discussion above (6), in order to obtain a formulation of the second-order equations in terms of a superpotential and hence a gradient flow. Had we retained the constant of integration, which would give a non-vanishing flux $\lambda \Omega_{(4)}$ for $F_{(4)}$ corresponding to M5-branes wrapping on 2-cycles in \mathbb{CP}^3 or $SU(3)/(U(1) \times U(1))$, it is again not clear how one would solve the second-order equations. We thank S.S. Gubser for raising this question about our procedure for obtaining gradient flows.

From this we can obtain first-order equations, given by

$$\begin{aligned} a' &= \frac{a^2 b^4 - 4b^6 - 2a^2 f_3^2}{8b^4 K}, & b' &= \frac{-2a^2 b^4 + (a^2 - 4b^2) f_3^2}{8a b^3 K}, \\ H' &= \frac{H (a^2 + 4b^2) f_3^2}{8a b^4 K}, & f_3' &= \frac{f_3 \left((12b^2 - a^2) b^4 - f_3^2 (a^2 + 20b^2) \right)}{8a b^4 K}, \end{aligned} \quad (42)$$

where $K \equiv \sqrt{b^4 - f_3^2}$ and a prime denotes a derivative with respect to the original ρ coordinate appearing in (31). Note that these first-order equations again imply an algebraic relation among the functions, analogous to (20). This time, we have

$$a^3 b H^6 f_3 = \kappa. \quad (43)$$

The equations can be solved by defining new quantities $\hat{a} \equiv H a$, $\hat{b} \equiv H b$ and $\hat{f}_3 \equiv H^2 f_3$. After manipulations analogous to those in section 2, we arrive at the general solution

$$ds_{11}^2 = H^2 dx^\mu dx_\mu + 12H^4 U^{-1} dr^2 + \frac{4}{3} r^2 H^{-2} U \nu_i^2 + r^2 H^{-2} \Sigma_i^2, \quad (44)$$

where

$$U \equiv 1 - \frac{\ell^3}{r^3}, \quad H = \left(1 - \frac{c^2}{r^{12} U^3} \right)^{-1/6}. \quad (45)$$

The function f_3 is given by

$$f_3 = \frac{c}{r^4 H^2 U^{3/2}}. \quad (46)$$

Note that the constant κ in (43) is related to c by $\kappa = 8c/(3\sqrt{3})$.

Thus we see that the general solution has two non-trivial integration constants, ℓ and c . The constant ℓ measures the scale size of the the gravitational instanton of the G_2 manifold, whilst the constant c measures the contribution from the 4-form field strength. Again, the solution is massless and carries no charge, and it can be thought of as a gravitational monopole involving the supergravity multiplet fields. Asymptotically, the solution becomes a product of four-dimensional Minkowski spacetime and the G_2 manifold with $\mathbb{R}^4 \times S^3$ topology, since the contribution to the metric is dominated at large r by the instanton contribution ℓ^4/r^4 , in comparison to the $F_{(4)}$ contribution which is of order c^2/r^{12} . At small distance, the solution has a naked singularity.

4 Supersymmetry of the M3-branes

Since the configurations that we have obtained in the previous two sections arise as the solutions of first-order systems of equations, it is natural to expect that they should be supersymmetric. In other words, one would expect that the first-order equations would

have the interpretation of being precisely the integrability conditions for supersymmetry. However, since they were not obtained by explicitly requiring supersymmetry, but rather by finding a superpotential for the Lagrangian formulation of the original bosonic supergravity equations of motion, the question of supersymmetry remains to be investigated.

4.1 Solutions in the \mathbb{R}^4 bundle over S^3 background

First, we shall study the supersymmetry for the solutions obtained in section 3, where the background metric in the transverse space is the \mathbb{R}^4 bundle over S^3 . It is a straightforward matter to calculate the spin connection for the metric (31) directly in eleven dimensions, and then to substitute this and the field strength ansatz (32) directly into the gravitino transformation rule

$$\delta\hat{\psi}_A = D_A \hat{\epsilon} - \frac{1}{288} F_{BCDE} \hat{\Gamma}_A{}^{BCDE} \hat{\epsilon} + \frac{1}{36} F_{ABCD} \hat{\Gamma}^{BCD} \hat{\epsilon}. \quad (47)$$

We make a standard $4 + 7$ decomposition of the Dirac matrices, as follows:

$$\hat{\Gamma}_\mu = \gamma_\mu \times \mathbf{1}, \quad \hat{\Gamma}_a = \gamma_5 \times \Gamma_a. \quad (48)$$

Substituting into (47), and examining first the world-volume directions μ , we find that a Killing spinor of the form $\hat{\epsilon} = \epsilon \times \eta$ must satisfy $\gamma_5 \epsilon = \pm \epsilon$. For the case of $\gamma_5 \epsilon = +\epsilon$ we find

$$\eta = g(b^2 + f_3)^{1/2} \eta_1 + g(b^2 - f_3)^{1/2} \eta_2, \quad (49)$$

where η_1 and η_2 are constant spinors in the transverse 7-space, satisfying the projection conditions

$$(\Gamma_1 + i\Gamma_4) \eta_2 = 0, \quad \Gamma_{23} \eta_1 = -\Gamma_{04} \eta_2, \quad (\Gamma_{26} - \Gamma_{35}) \eta_1 = -2i\Gamma_{04} \eta_2. \quad (50)$$

Note that these conditions uniquely determine η_1 and η_2 , up to an overall scale. For the case where $\gamma_5 \epsilon = -\epsilon$, the associated spinor η is given again by (49), but with f_3 replaced by $-f_3$. The dependence of the overall function g on the coordinates of the transverse space is undetermined by the μ components of $\delta\hat{\psi}_A = 0$.

The components of $\delta\hat{\psi}_A = 0$ lying in the directions $A = a$ of the transverse space will now determine the dependence of g on the transverse coordinates. It is easiest first to examine the radial direction (i.e. the “0” direction), which determines the radial dependence of the Killing spinor. Then, by looking at the remaining transverse directions, we find that the Killing spinor has no dependence on the angular coordinates of the two 3-spheres. The

conclusion is that the function g in (49) is given by

$$g = b^{-1} H^{1/2}. \quad (51)$$

Thus the first-order equations (37) (where f_1 and f_2 are given by (42) with $\lambda = 0$) are precisely the integrability conditions for the existence of a spinor $\hat{\epsilon}$ satisfying $\delta\hat{\psi}_A = 0$ in (47). Since we have 2 solutions (corresponding to two spinors ϵ in the M3-brane world-volume) for each of the cases $\gamma_5 \epsilon = \pm\epsilon$, the general Killing spinor has four real solutions, corresponding to $N = 1$ supersymmetry on the world-volume of the M3-brane. Of course if the constant c is set to zero, so that the 4-form is turned off, these Killing spinors reduce to the usual ones in the product of four-dimensional Minkowski spacetime and the Ricci-flat metric of G_2 holonomy.

4.2 Solutions in the \mathbb{R}^3 bundle over S^4 background

Here, we repeat the analysis of the supersymmetry transformations in the case of the \mathbb{R}^3 bundle over S^4 background, for which the M3-brane solution was constructed in section 2.

Again we begin by considering the components of $\delta\hat{\psi}_A = 0$ lying in the world-volume of the 3-brane. From $\delta\hat{\psi}_\mu = 0$ we deduce that a Killing spinor of the form $\hat{\epsilon} = \epsilon \times \eta$ will be given by $\gamma_5 \epsilon = \pm\epsilon$, and for the case $\gamma_5 \epsilon = +\epsilon$ we have

$$\eta = g P \left[(K - f)^{1/2} \eta_1 + (K + f)^{1/2} \eta_2 \right], \quad (52)$$

where K is given by (19), P is given by

$$\begin{aligned} P \equiv & -\frac{\sin \frac{1}{2}\theta}{2ab^2} \left[(K \cos \frac{1}{2}\phi - i f \sin \frac{1}{2}\phi) \Gamma_{01} + (K \cos \frac{1}{2}\phi + i f \sin \frac{1}{2}\phi) \Gamma_{02} \right] \\ & + \cos \frac{1}{2}\theta (\cos \frac{1}{2}\phi + \sin \frac{1}{2}\phi \Gamma_{12}), \end{aligned} \quad (53)$$

and the constant 8-component spinors η_1 and η_2 are uniquely specified (up to scale) by the projections

$$\begin{aligned} (\Gamma_0 - \Gamma_{3456}) \eta_2 &= 0, & (\Gamma_{34} + \Gamma_{56} - 2\Gamma_{012}) \eta_2 &= 0, \\ 4\eta_1 &= (\Gamma_{135} - \Gamma_{146} + \Gamma_{236} + \Gamma_{245}) \eta_2. \end{aligned} \quad (54)$$

Here the explicit indices 1 and 2 on the Dirac matrices refer to the two directions on the S^2 fibres, as in (3), whilst the indices 3, 4, 5 and 6 refer to the directions in the S^4 base. The index 0 refers to the radial direction. For the case when $\gamma_5 \epsilon = -\epsilon$, the associated spinor η in the transverse space will be given again by (52), but now with f sent to $-f$ in (52) and in (53).

The dependence of the overall prefactor g on the coordinates of the transverse space is not determined by $\delta\hat{\psi}_\mu = 0$ in the world-volume directions. (We have, however, made a convenient choice of θ and ϕ dependent overall factors, in anticipation of subsequent results.)

From the radial component $\delta\hat{\psi}_0 = 0$ in the transverse space, we can again determine the radial dependence of the function g , finding

$$g = \frac{\hat{g}}{H b a^{1/2}}, \quad (55)$$

where \hat{g} depends only on the angular coordinates of the transverse space. Examination of $\delta\hat{\psi}_A = 0$ in the S^2 directions then implies that \hat{g} is independent of these two coordinates. Finally, the components in the S^4 directions determine that \hat{g} has the dependence associated with the singlet fermion zero-mode in the Yang-Mills instanton background (as in [29]).

We have seen that again, the first-order system of equations for this 3-brane in the background of the \mathbb{R}^3 bundle over S^4 have turned out to be precisely the integrability conditions for the existence of a Killing spinor. There are in total four real solutions, implying $N = 1$ supersymmetry on the world-volume of the M3-brane. As in the previous example, if the field strength in the solution is taken to zero, by setting the constant $c = 0$, the Killing spinor reduces to the standard one in the vacuum of four-dimensional Minkowski spacetime times the Ricci-flat metric on the \mathbb{R}^3 bundle over S^4 .

5 Dual formulations and phase transitions

In a standard massive BPS p -brane solution, the charge Q arises as a constant prefactor in the field strength supporting the solution, and Q also appears linearly in the harmonic function H in the p -brane metric. By contrast, in our massless M3-brane solutions the analogous constant c that arises as the prefactor in the expressions for the 4-form field appears quadratically in the metrics (25) and (44). This means that the metrics would continue to be real if we were to send $c \rightarrow ic$. The same would also be true of the reduced metrics in $D = 7$ that formed the starting-points of our derivations in sections 2 and 3.

Of course sending $c \rightarrow ic$ would imply that the 4-form field strength would become imaginary. However, it should be recalled that our original 7-dimensional starting point was in the Euclidean-signatured theory obtained by dimensional reduction on the world-volume of the M3-brane. In Euclidean signature, if the 4-form field strength $F_{(4)}$ is dualised to a 3-form $F_{(3)}$, then its kinetic term in the $D = 7$ Lagrangian will undergo the replacement

$$-\frac{1}{48} e^{\sqrt{8/5}\phi} F_{(4)}^2 \rightarrow +\frac{1}{12} e^{-\sqrt{8/5}\phi} F_{(3)}^2. \quad (56)$$

This change of sign of the kinetic term, which is generic to all dualisations in Euclidean signature, indicates that we could achieve the same effect as sending $c \rightarrow ic$ by instead using a *real* 3-form field in $D = 7$, but with the canonical $-\frac{1}{12} e^{-\sqrt{8/5}\phi} F_{(3)}^2$ kinetic term instead of the sign-reversed one in (56) that arose by dualising the 4-form.

The upshot of the above discussion is that we can obtain real solutions in $D = 7$ that are just like those in sections 2 and 3, but for the opposite sign of c^2 . These will be solutions of the equations coming from the seven-dimensional Lagrangian

$$e^{-1} \tilde{\mathcal{L}}_7 = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12} e^{-\sqrt{\frac{8}{5}}\phi} F_{(3)}^2. \quad (57)$$

The expression for $F_{(3)}$ for each solution will be given by $F_{(3)} = -i * F_{(4)}$, where $F_{(4)}$ is the corresponding expression given in section 2 or 3. The i factor in this relation between $F_{(3)}$ and $F_{(4)}$ is precisely removed by the i factor that we acquire upon sending $c \rightarrow ic$.

A difference now arises when we consider the higher-dimensional origin of the seven-dimensional Lagrangian. We viewed (8) in sections 2 and 3 as coming from the Kaluza-Klein reduction of $D = 11$ supergravity on the world-volume of the M3-brane. Instead, we should now view (57) as coming from the Kaluza-Klein reduction of type IIA, type IIB or type I supergravity on the world-volume of a 2-brane. In other words, we obtain the 3-form in $D = 7$ as the direct world-volume reduction of a 3-form in $D = 10$. Accordingly, we can then lift the $D = 7$ solutions of sections 2 and 3, after sending $c \rightarrow ic$, to real solutions of ten-dimensional supergravity. Thus there is a phase transition from one type of brane to another, when we change the modulus parameter c of the solution from real to imaginary.

For the case of the S^2 bundle over S^4 in section 2, we find that the corresponding massless 2-brane solution in $D = 10$ is given by

$$ds_{10}^2 = H^{-3/2} dx^\mu dx_\mu + 2H^{-9/2} U^{-1} dr^2 + \frac{1}{2} r^2 H^{3/2} U D\mu^i D\mu^i + r^2 H^{3/2} d\Omega_4^2, \quad (58)$$

where

$$U = 1 - \frac{\ell^4}{r^4}, \quad H = \left(1 - \frac{c^2}{2r^{12}U^2}\right)^{1/6}. \quad (59)$$

For the S^3 bundle over S^3 of section 3, the corresponding massless 2-brane solution in $D = 10$ is given by

$$ds_{10}^2 = H^{-3/2} dx^\mu dx_\mu + 12H^{13/2} U^{-1} dr^2 + \frac{4}{3} r^2 H^{1/2} U \nu_i^2 + r^2 H^{1/2} \Sigma_i^2, \quad (60)$$

where

$$U \equiv 1 - \frac{\ell^3}{r^3}, \quad H = \left(1 + \frac{c^2}{r^{12}U^3}\right)^{-1/6}. \quad (61)$$

We have written the solutions that come from reducing the NS-NS 3-form of the ten-dimensional supergravity. Of course in the case of type IIB we could instead use the R-R 3-form, in which case the lifted solutions in $D = 10$ would simply be the S-duals of those we have just presented.

One can also, of course, further lift the above configurations, if viewed as solutions of type IIA supergravity, to $D = 11$. For the case corresponding to the S^2 bundle over S^4 we then find

$$ds_{11}^2 = H^{-2} dx^\mu dx_\mu + H^4 dz^2 + 2H^{-5} U^{-1} dr^2 + \frac{1}{2} r^2 H U D\mu^i D\mu^i + r^2 H d\Omega_4^2, \quad (62)$$

where z is the eleventh coordinate, and U and H are again given by (59). For the case corresponding to the S^3 bundle over S^3 , we find

$$ds_{11}^2 = H^{-2} dx^\mu dx_\mu + H^4 dz^2 + 12H^6 U^{-1} dr^2 + \frac{4}{3} r^2 U \nu_i^2 + r^2 \Sigma_i^2, \quad (63)$$

where U and H are given by (61).

6 $D = 7$ 3-brane and S^2 -wrapped M5-brane

It is of interest also to study more general 3-brane configurations in M-theory. Another natural candidate is an M5-brane wrapped around a supersymmetric 2-cycle. M5-branes wrapped on supersymmetric cycles have been discussed previously [30, 31]. Typically, they admit solutions of the form $\text{AdS}_d \times H_{7-d}$, where H_n denotes the n -dimensional hyperbolic space. In this section, we shall consider an M5-brane wrapped around a 2-sphere. The solution can be obtained by looking first at gauged supergravity in $D = 7$.

6.1 $D = 7$ AdS_7 3-brane

Consider the $D = 7$, $N = 2$ gauged supergravity, whose bosonic Lagrangian is

$$\begin{aligned} \mathcal{L}_7 = & R * \mathbf{1} - \frac{1}{2} * d\phi \wedge d\phi - U * \mathbf{1} - \frac{1}{2} e^{-\frac{4}{\sqrt{10}}\phi} * F_{(4)} \wedge F_{(4)} \\ & - \frac{1}{2} e^{\frac{2}{\sqrt{10}}\phi} * F_{(2)} \wedge F_{(2)} + \frac{1}{2} F_{(2)}^i \wedge F_{(2)}^i \wedge A_{(3)} - \frac{1}{2\sqrt{2}} g F_{(4)} \wedge A_{(3)}. \end{aligned} \quad (64)$$

where $F_{(4)} = dA_{(3)}$ and U is the scalar potential in the $D = 7$ gauged supergravity,

$$U = g^2 \left(\frac{1}{4} e^{\frac{8}{\sqrt{10}}\phi} - 2e^{\frac{3}{\sqrt{10}}\phi} - 2e^{-\frac{2}{\sqrt{10}}\phi} \right). \quad (65)$$

In addition, the 4-form satisfies the first-order odd-dimensional self-duality equation

$$e^{-\frac{4}{\sqrt{10}}\phi} * F_{(4)} = -\frac{1}{\sqrt{2}} g A_{(3)} + \frac{1}{2} \omega_{(3)}. \quad (66)$$

Here, we have $\omega_{(3)} \equiv A_{(1)}^i \wedge F_{(2)}^i - \frac{1}{6}g \epsilon_{ijk} A_{(1)}^i \wedge A_{(1)}^j \wedge A_{(1)}^k$. Domain wall and AdS₇ black hole solutions in this theory have been constructed [32, 33, 34], which can be viewed after lifting back to M-theory as distributed or rotating M5-branes respectively.

Here we consider a 3-brane configuration, which is supported by one component of the $SU(2)$ Yang-Mills gauge fields. We take the ansatz to be

$$ds_7^2 = e^{2A} dx^\mu \wedge dx_\mu + e^{2B} (dr^2 + d\Omega_2^2), \quad (67)$$

$$F_{(2)}^3 = \lambda \Omega_2, \quad F_{(4)} = 0. \quad (68)$$

The resulting equations of motion can be derived from the Lagrangian $L = T - V$, where

$$\begin{aligned} T &= -12\dot{A}^2 - 16\dot{A}\dot{B} - 2\dot{B}^2 + \frac{1}{2}\dot{\phi}^2, \\ V &= e^{8A+2B} \left(2 - e^{2B} U - \frac{1}{2}\lambda^2 e^{-2B+\frac{2}{\sqrt{10}}\phi} \right), \end{aligned} \quad (69)$$

together with the constraint $T + V = 0$. Here the dot denotes a derivative with respect to η , defined by $d\eta = e^{8A+2B} dr$.

We find that V can be derived from a superpotential W , provided that $g\lambda = 1$. It is given by

$$W = 2\sqrt{2}g e^{4A+2B-\frac{1}{\sqrt{10}}\phi} + \sqrt{2}g^{-1} e^{4A+\frac{1}{\sqrt{10}}\phi} + \frac{1}{\sqrt{2}}g e^{4A+2B+\frac{4}{\sqrt{10}}\phi}. \quad (70)$$

The associated first-order equations, after setting $g = 1$ without loss of generality, are given by

$$\frac{a'}{a} = -b^2 f^4 - \frac{4b^2}{f} + 2f, \quad \frac{b'}{b} = -b^2 f^4 - \frac{4b^2}{f} - 8f, \quad \frac{f'}{f} = 4b^2 f^4 - \frac{4b^2}{f} + 2f, \quad (71)$$

where $a = e^A$, $b = e^B$, $f = e^{\frac{1}{\sqrt{10}}\phi}$, and a prime here denotes a derivative with respect to ρ , which is defined by $d\rho = \frac{1}{10\sqrt{2}}e^{-B} dr$.

It is not clear how to solve these first-order equations analytically, but the general behaviour of the solutions to the gradient flow can nevertheless be analysed in terms of a phase-plane diagram. From (71), we can plot the 2-dimensional vector (b', f') , and it shows that the solution flows from $(b \rightarrow \infty, f \rightarrow 1)$ to $(b \rightarrow 0, f \rightarrow \infty)$. (See figure 1. Note that f is always non-negative.)

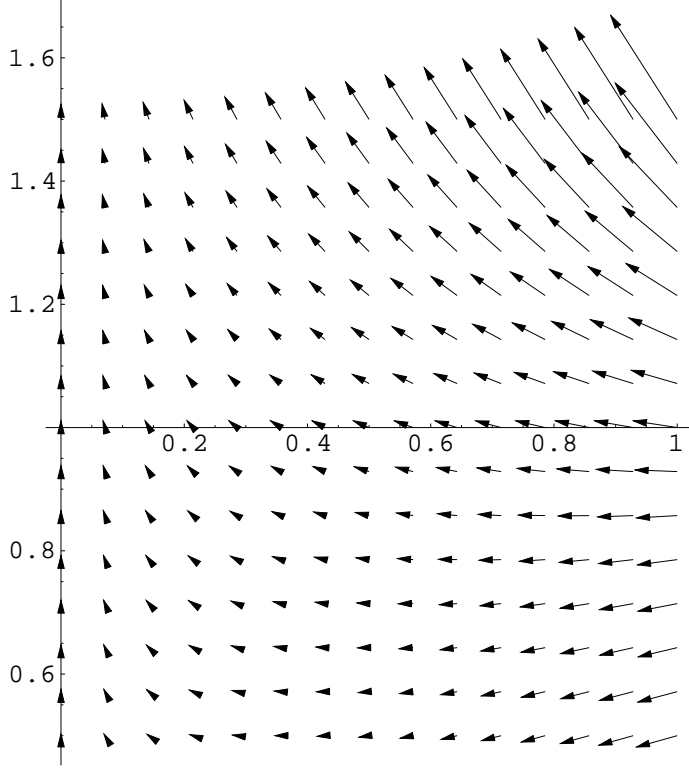


Figure 1. The flow defined by the 2-vector (b', f') . The abscissa is b and the ordinate is f .

It suffices to analyse the solution in the regions $(b \rightarrow \infty, f \rightarrow 1)$ and $(b \rightarrow 0, f \rightarrow \infty)$. Consider first the behaviour when $f \rightarrow 1$. In this case, the approximate form of the solution is

$$\begin{aligned} a^2 &\sim \frac{1}{10\rho} + \frac{6}{5} + \frac{181\rho}{30} + \dots, & b^2 &\sim \frac{1}{10\rho} - \frac{4}{5} + \frac{11\rho}{15} + \dots, \\ f &\sim 1 - 2\rho + \rho^2 \left(\frac{638}{21} + 40 \log \rho \right) - 1120\rho^3 \log \rho + \dots, \end{aligned} \quad (72)$$

up to the first few orders in ρ . It is easy to see that $r \sim \sqrt{\rho} \rightarrow 0$, and so the metric approaches

$$ds_7^2 \sim \frac{1}{r^2} (dx^\mu dx_\mu + d\Omega_2^2 + dr^2). \quad (73)$$

Since r tends to zero here, this describes the large-distance asymptotic region.

Now consider the behaviour when $f \rightarrow \infty$, with b approaching zero. In this case, we have

$$a^2 \sim (1 - \rho)^{1/7} \sim b^2, \quad f \sim \frac{1}{2}(1 - \rho)^{-2/7}. \quad (74)$$

with ρ tending to 1 from below. The metric then has the form

$$ds_7^2 = (r - r_0)^{2/15} (dx^\mu dx_\mu + d\Omega_2^2 + dr^2). \quad (75)$$

Since $r \rightarrow r_0$ in this case, it clearly corresponds to the region at small proper distance. The solution at $r = 0$ would be singular, but it is also a horizon

We can dimensionally reduce the solution on $d\Omega_2^2$, to obtain a domain wall in $D = 5$, of the form

$$ds_5^2 = e^{2C} (dx^\mu dx_\mu + dr^2). \quad (76)$$

The radial coordinate r runs from $r = 0$, which is the AdS₅ horizon, to $r = r_0$, which is a null singularity. The conformal factor in these two regions is given by

$$\begin{aligned} r \rightarrow 0 : \quad e^{2C} &\sim \frac{1}{r^{10/3}}, & V &\sim \frac{35}{4r^2} \\ r \rightarrow r_0 : \quad e^{2C} &\sim (r - r_0)^{2/9}, & V &\sim \frac{1}{12(r - r_0)^2}. \end{aligned} \quad (77)$$

Thus the system has a discrete spectrum, indicating confinement.

6.2 Lifting to S^2 -wrapped M5-brane

The consistent S^4 reduction of eleven-dimensional supergravity was obtained in [35, 36, 37]. Using the explicit reduction ansatz given in [36], we can lift the above solution to give an S^2 -wrapped M5-brane in $D = 11$, with

$$\begin{aligned} ds_{11}^2 &= \Delta^{1/3} \left(a^2 dx^\mu dx_\mu + b^2 (dr^2 + d\Omega_2^2) \right) + 2f^{-1/3} \Delta^{1/3} d\xi^2 + \frac{1}{2} \Delta^{-2/3} f \cos^2 \xi (\sigma^2 + d\tilde{\Omega}_2^2), \\ A_{(3)} &= \frac{1}{\sqrt{2}} \sin \xi \sigma \left(\tilde{\Omega}_{(2)} - \Omega_{(2)} + \frac{1}{2} \cos^2 \xi \Delta^{-1} f^4 \tilde{\Omega}_{(2)} \right). \end{aligned} \quad (78)$$

Here

$$\Delta = f^4 \sin^2 \xi + f^{-1} \cos^2 \xi, \quad d\sigma = \Omega_{(2)} + \tilde{\Omega}_{(2)}. \quad (79)$$

In the asymptotic region at large proper distance, the metric becomes

$$ds_{11}^2 \sim \frac{1}{r^2} (dx^\mu dx_\mu + d\Omega_2^2 + dr^2) + 2d\xi^2 + \frac{1}{2} \cos^2 \xi (\sigma^2 + d\tilde{\Omega}_2^2), \quad (80)$$

where $r \rightarrow 0$. Note that the 4-form field strength $F_{(4)} = dA_{(3)}$ has a term

$$F_{(4)} = \frac{1}{2\sqrt{2}} \Delta^{-1} f^4 \cos^3(\xi) d\xi \wedge \sigma \wedge \tilde{\Omega}_2 + \cdots, \quad (81)$$

implying that this 3-brane configuration has non-vanishing M5-brane charge.

6.3 General solutions

Although we have not obtained the general solution explicitly, we can nevertheless show that the first-order equations (71) can be reduced to a single non-linear first-order differential equation. Defining $X \equiv b/f$, $Y \equiv b f^4$, and $dt = 5f d\rho$, we have

$$\frac{a'}{a} = \frac{1}{5}(-XY - 4X^2 + 2), \quad \frac{X'}{X} = -XY - 2, \quad \frac{Y'}{Y} = 3XY - 4X^2. \quad (82)$$

The first equation gives a , once X and Y have been found using the remaining equations. The second equation may be solved for Y , and substituted into the third. This gives

$$X X'' + X'^2 + 4X^3 X' + 10X X' + 8X^4 + 12X^2 = 0. \quad (83)$$

Now let $v \equiv X'$, so that $X'' = v' = dv/dX$, $dX/d\rho = v$, and hence (83) becomes

$$v X \frac{dv}{dX} + v^2 + 4v X^3 + 10v X + 8X^4 + 12X^2 = 0. \quad (84)$$

A further change of variable from v to w , defined by $w \equiv \frac{1}{2}v X$, then gives

$$X^{-1} w \frac{dw}{dX} + 2w X^2 + 5w + 2X^4 + 3X^2 = 0. \quad (85)$$

Finally, we let $z \equiv X^2 + \frac{5}{2}$. This transforms (85) into

$$w \frac{dw}{dz} + z w + \frac{1}{2}(z - 1)(2z - 5) = 0. \quad (86)$$

This is a particular case of Abel's equation, but unfortunately it appears to be difficult to obtain the solution in closed form.

6.4 Non-abelian solutions in $D = 7$ supergravity

So far we have made use only of a $U(1)$ subgroup of the $SU(2)$ gauge fields. It is possible also to turn on the full $SU(2)$ gauge fields, with the ansatz

$$A_{(1)}^1 = v \sin \theta d\phi, \quad A_{(1)}^2 = -v d\theta, \quad A_{(1)}^3 = \cos \theta d\phi, \quad (87)$$

where v is a function of r , and (θ, ϕ) are the coordinates on the 2-spheres foliating the transverse 3-space. (This ansatz was used, for example, in [38].) The Hamiltonian $H = T + V$ for this case is given by

$$\begin{aligned} T &= -12A'^2 - 16A' B' - 2B'^2 + \frac{1}{2}\phi'^2 + e^{-2B + \frac{2}{\sqrt{10}}\phi} v'^2, \\ V &= e^{8A + 2B} \left(2 - e^{2B} U - \frac{1}{2} e^{-2B + \frac{2}{\sqrt{10}}\phi} (v^2 - 1)^2 \right), \end{aligned} \quad (88)$$

where U is the scalar potential in $D = 7$ gauged supergravity, as given in (65).

We have not found a superpotential for this system. The earlier $U(1)$ result corresponds to $v = 0$. There is a singular scaling limit in which the first two terms in the scalar potential U in (65) vanish, and then the theory can be viewed as the S^3 reduction of $\mathcal{N} = 1$, $D = 10$ supergravity [39]. A supersymmetric 3-brane with $SU(2)$ Yang-Mills fields does then exist, and it is non-singular [40]. The solution is the lifting to $D = 10$ of the $SU(2)$ black hole constructed in [38]. The superpotential for this system was obtained in [27].

7 Conclusions

In the context of four-dimensional field theories, it is of considerable interest to construct 3-brane configurations in M-theory. One class of such solutions has been obtained by wrapping M5-branes on a certain supersymmetric two cycles, such as Riemannian surfaces [30]. These solutions are deformations of an $\text{AdS}_5 \times H^2 \times S^4$ vacuum, where H^2 is the hyperbolic plane.

In this paper, we have constructed two new types of 3-brane configuration. In the first type, we exploit the fact that the transverse space of the 3-brane in $D = 11$ is seven-dimensional, and that there exist non-trivial seven-dimensional Ricci-flat manifolds with G_2 holonomy. It has been proposed that compactifications of M-theory on G_2 are related to $\mathcal{N} = 1$, $D = 4$ Yang-Mills theory [8, 9, 10]. Three explicit complete non-compact manifolds of G_2 holonomy are currently known [16, 17], and they can be used to smooth out the singularities of compact G_2 orbifolds. Each of them admits a harmonic 4-form [18, 19], which suggests the possibility of turning on the 4-form field strength of $D = 11$ supergravity in solutions that correspond to deformations of four-dimensional Minkowski space times a Ricci-flat G_2 manifold. We have indeed managed to obtain exact solutions of this type, which can be viewed as 3-branes in M-theory.

The general solutions contain two continuous parameters; ℓ , which measures the size of the gravitational instanton, and c , which measures the strength of the 4-form. The solutions are massless, and carry no 4-form charge. In common with all other known massless brane solutions, there are naked singularities at short distance. At large distance the solution approaches the product of four-dimensional Minkowski spacetime and the original Ricci-flat G_2 manifold. The solution can be viewed as a supergravitational monopole, involving both the metric and the 4-form in the supergravity multiplet.

We obtained the solutions by deriving first-order equations from a superpotential, and we showed that these are precisely the integrability conditions for the existence of a Killing

spinor. Thus the two M3-brane solutions that we have constructed in this paper are supersymmetric.

It is interesting to observe that although the harmonic 4-forms in the undeformed manifolds of G_2 holonomy, on the \mathbb{R}^3 bundle over S^4 and the \mathbb{R}^4 bundle over S^3 , have quite different properties (the former being L^2 normalisable whilst the latter is not), the corresponding deformed 3-brane solutions in sections 2 and 3 have very similar qualitative behaviour. In contrast, the properties of the metrics for the fractional D2-brane [19] and NS-NS 2-brane [18], which make use of these same two Ricci-flat metrics, are significantly different.

We also obtained M3-brane solutions of a different kind, by lifting 3-brane solutions in $D = 7$ gauged supergravity back to $D = 11$. They carry magnetic 4-form charge, and can be viewed as M5-branes wrapped on S^2 .

Acknowledgement

We are grateful to Mike Duff, Gary Gibbons, James Liu and Jianxin Lu for useful discussions. C.N.P. is grateful to the Michigan Center for Theoretical Physics for hospitality during the completion of this work.

Note added

In an earlier version of this paper it was claimed that the M3-brane solutions were not supersymmetric, but instead were “pseudo-supersymmetric” with respect to a modified $D = 11$ supersymmetry transformation rule. This incorrect conclusion resulted from a systematic error in a computer program that we used for calculating the Killing spinors. We are grateful to Jim Liu for calculations that encouraged us to recheck the computer programs and discover the error.

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